Kurtosis

For any distribution, the kurtosis (sometimes called the excess kurtosis) is defined as

\[ \gamma_2 = \varphi_4 - 3 \]

(\text{old notation} = \beta_2)

For a unimodal, symmetric distribution,
- a positive kurtosis means “heavy tails” and a more peaked center compared to a normal distribution
- a negative kurtosis means “light tails” and a more spread center compared to a normal distribution

Kurtosis Examples

- For the Student’s t distribution, the excess kurtosis is
  \[ \gamma_2 = \frac{6}{\text{DF} - 4} \]
  for DF > 4 (for DF ≤ 4 the kurtosis is infinite)
- For a uniform distribution, \( \gamma_2 = -\frac{6}{5} \)

One Impact of Excess Kurtosis

- For a normal distribution, the sample variance will have an expected value of \( \sigma^2 \), and a variance of
  \[ \text{var}(\hat{\sigma}^2) = \frac{2\sigma^4}{n - 1} \]
- For a distribution with excess kurtosis \( \gamma_2 \)
  \[ \text{var}(\hat{\sigma}^2) = \frac{2\sigma^4}{n - 1} \left( 1 + \frac{n - 1}{2n\gamma_2} \right) \]

Sample Kurtosis

- For a sample of size \( n \), the sample kurtosis is
  \[ g_2 = \frac{1}{n} \sum_{i=1}^{n} (x - \bar{x})^4 \]
  \[ 3 - \frac{1}{n} \sum_{i=1}^{n} (x - \bar{x})^2 \]
  \[ \text{for large } n, \text{ the sampling distribution of } g_2 \text{ approaches Normal with mean 0 and variance } 24/n \]
  \[ \text{for small samples, this estimator is biased} \]

An unbiased estimator of the sample excess kurtosis is

\[ G_2 = \frac{n - 1}{(n - 2)(n - 3)} \left[ (n + 1)g_2 + 6 \right] \]

Standard Error:

\[ SE(G_2) = 2SE(G_1) \]

Sample Kurtosis Test

- Only perform the kurtosis test if the skewness test fails to reject the null hypothesis
- Null hypothesis: $\gamma_2 = 0$ (Normal distribution)
- Test statistic: $G_2 / SE(G_2)$ is approximately standard Normal for $n > 20$
  - We generally perform a two-tailed test
- If the test statistic $G_2 / SE(G_2)$ is beyond the critical $z$-value for our significance level we reject the null hypothesis that the distribution is Normal

Jarque-Berra Test

- The tests for skewness and kurtosis can be combined into one
  $\left( \frac{G_1}{SE(G_1)} \right)^2 + \left( \frac{G_2}{SE(G_2)} \right)^2 \sim \chi^2(2)$
  - For example, the 95% ($\alpha = 0.05$) critical value for $\chi^2(2)$ is 5.99 and the 99% critical value is 9.21

Impact of Outliers

- Both the Skewness test and the Kurtosis test are very sensitive outlier detectors
  - One outlier will make the distribution appear skewed
  - Two symmetric outliers will make the tails appear heavy
- More on outlier detection in the next lectures

Lecture 14: What have we learned?

- How is kurtosis defined?
- For positive excess Kurtosis, what is the shape of the pdf?
- Be able to test a sample data set for excess kurtosis. What test statistic is used? What is its sampling distribution?