Lecture 79
Propagation of Uncertainty

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The Measurement Model

- Consider a measurement model
  \[ y = f(x_1, x_2, \ldots) \]
- Some of the inputs represent the thing we want to measure, but some are nuisance variables
  - Ex: Measure length with a steel ruler; temperature is a nuisance variable
- How do variations in the \( x_i \) propagate to variations in \( y? \)
  - Called the propagation of uncertainty

Propagation of Uncertainty

- There are three common ways to propagate uncertainty from inputs to an output
  - Propagation of pdfs (often difficult to do)
  - Taylor Series (most common, but easy to do wrong)
  - Monte Carlo simulations (often best approach for complex measurement models)

Propagation of PDFs (1D example)

- Consider \( Y = f(X) \) where \( X \) is a random variable with known pdf \( P_X(x) \). What is the pdf of \( Y \)?

  \[ P_Y(y) = \frac{dx}{dy} P_X(x) \]

  where \( x = f^{-1}(y) \)
- Note: \( f^{-1}(y) \) can have multiple roots

Taylor Series Approach

- Expand the function as a Taylor series
  \[ f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \]
- Let \( \Delta y = f(x) - f(a) \), \( \Delta x = x - a = \alpha \)
- \( \Delta y = \frac{df}{dx} \Delta x + \frac{1}{2!} \frac{d^2f}{dx^2} \Delta x^2 + \cdots + \frac{1}{n!} \frac{d^n f}{dx^n} \Delta x^n \)
- For \( K \) different input variables,

  \[ \Delta y = \sum_{j=1}^{K} \left( \frac{1}{1!} \sum_{k=1}^{n_k} \frac{\partial f}{\partial x_k} \Delta x_k \right)^{j} \]
Taylor Series Approach

- For two input variables $x_1$ and $x_2$,
  \[ \sigma_y^2 = \left( \frac{df}{dx_1} \right)^2 \sigma_{x_1}^2 + \left( \frac{df}{dx_2} \right)^2 \sigma_{x_2}^2 + 2 \left( \frac{df}{dx_1} \right) \left( \frac{df}{dx_2} \right) \text{cov}(x_1, x_2) + \ldots \]
  - When the input errors are small and the slopes are not, we can ignore higher order terms
- When $x_1$ and $x_2$ are independent,
  \[ \sigma_y^2 = \left( \frac{df}{dx_1} \right)^2 \sigma_{x_1}^2 + \left( \frac{df}{dx_2} \right)^2 \sigma_{x_2}^2 \]

Some Examples

- Assume small, independent errors
  - Keep only linear terms, ignore covariance
- $y = ax_1 + bx_2$, $\sigma_y^2 = a^2 \sigma_{x_1}^2 + b^2 \sigma_{x_2}^2$
- $y = x_1 x_2$, assuming $\bar{x}_i \gg \sigma_{x_1}$
  \[ \left( \frac{\sigma_y}{\bar{y}} \right)^2 = \left( \frac{\sigma_{x_1}}{\bar{x}_1} \right)^2 + \left( \frac{\sigma_{x_2}}{\bar{x}_2} \right)^2 \]
- $y = \ln(ax)$, $\sigma_y^2 = a^2 / \bar{x}^2$

Using a Micrometer

Images from Wikimedia Commons

Micrometer Example

- Sources of uncertainty:
  - Scale (calibration) error: 3 μm, assumed to be uniformly distributed ($s = \text{range}/\sqrt{3} = 1.73 \mu m$)
  - Zero point error: 2 μm ($s = 1.15 \mu m$)
  - Anvil parallelism: $s = 0.58 \mu m$
  - Temperature different between micrometer and object: 3 °C, leading to $s = 0.61 \mu m$
  - Measurement repeatability: $s = 2 \mu m$
- Combined standard uncertainty = 3.0 μm

Lecture 79: What have we learned?

- What are the major difficulties of Bayesian regression?
- Explain how frequentist and Bayesian regression concepts can merge in a "hybrid" form
- Explain how this "hybrid" form relates to ridge regression